Analytic Proof of the Attractors of a Class of Cellular Automaton

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Abstract

In this work we provide analytic results of infinite one-dimensional cellular automaton(**CA**). By realizing symbolic products, we investigate a subclass of infinite **CA** and prove analytically that within this subclass the only allowed attractors are homogenous, steady and periodic states for arbitrary initial configuration. Our method also provide exact enumeration of these attractors and it is shown explicitly in a particular model.

1 Introduction

Cellular automata(CA) are simple mathematical models which can generate complex dynamical phenomena. In principle, **CA** are discrete dynamical systems defined on a discrete lattice. The state of each site at any time $\boldsymbol{t}$ is in one of the $\boldsymbol{g}$ states. The interaction between sites is according to given local rules and the system evolves synchronously in discrete time steps. Cellular automaton was introduced by von Neumann in order to address the problems of self-reproduction and evolution [1]. However **CA** has not attracted much attention until John Conway introduced the game of lifes [2] around 1970 by using **CA**. About twenty years ago, Stephen Wolfram introduced cellular automaton to the physics community as models of complex dynamical systems [3] and a new approach to the parallel computing scheme(For a review see [4]). The interest in **CA**'s potential application continues growing. In fact cellular automata have been used to simulate, for example, solar flares [5], fluid dynamics [6], crystal growth [7], traffic flow [8] and galaxy formation [9].

Based on large number of numerical studies, Wolfram has suggested that **CA** can be classified into four classes. Cellular automata within each class has the same qualitative behavior. Starting from almost all initial conditions, trajectories of **CA** become concentrated onto attractors, the four classes can then be characterized by their attractors. According to Wolfram's classification, the class 1, 2 and 3 are roughly corresponding to the limit points, limit cycles and chaotic attractors in continuous dynamical systems respectively. More precisely their respective long time limits are : (1) spatially homogeneous state, (2) fixed (steady) or periodic structure and (3) chaotic pattern throughout space. The fourth class of **CA** behaves in a much more complicated manner and was conjectured by Wolfram as capable for performing universal computation.

Due to the complexity of cellular automata, numerical computation of time evolution becomes the main scheme in **CA** studies. In [3] some statistical approaches were introduced to characterize quantitatively the patterns generated by **CA** evolution. Statistical quantities such as entropies and dimensions can only provide average properties of cellular automata. However, exact results are always needed in more elaborate discussions . To make this point more explicit let us consider the effects of noise on **CA**. Due to the fact that all physical systems are coupled to noisy environment, it is interesting to know if the above classification of the attractors of **CA** remains intact under the

influence of noise. But this question can only be answered by knowing the deterministic **CA** exactly, and as a result the exact evaluation of the attractor without noise is called for. Unfortunately such exact calculations only exists for a few cases (A good review can be found in [10]). For example, the GKL automaton was proposed in the 80' [11] and surprisingly the detail analytic proof was completed about a decade later [12]. For finite **CA** with majority rules, exact results have been obtained on finite lattice[13]. It is shown in [13] that the attractors are either steady states or periodic states of period two. It is important to note that the result of [13] contains the majority rules as a special case. However, for a system of finite lattice, the configuration space is finite and can only contain periodic final states. Moreover, for infinite lattice, it is also known that the attractors are steady states if the initial state is finite[14](The finite initial condition will be defined in the next section.). Since the above results are only true for special configurations, the analysis for the attractors of infinite **CA** with arbitrary initial configuration is still lacking. It is the purpose of this work to fill this gap.

In this work we provide the analytic results of infinite one-dimensional cellular automaton. We investigate a subclass of infinite **CA** and prove analytically that within this subclass the chaotic and complex structures are excluded. As a result, the only allowed attractors are homogenous, steady and periodic states. The main tool for proving the existence of steady states is the construction of a symbolic product among the fundamental blocks which will be introduced later. In section 2, we briefly introduce notations and formulation of **CA**. In section 3 the product rules are given and several lemmas are proved to set the stage for the proof of the main result. The allowed attractors are shown explicitly for the case of $k = 2$ in section 4. Finally, a brief discussion is given in section 5.

2 CA review and notations

A cellular automaton is a spatial lattice of N sites where N can be finite or infinite. The state of each site can be in one of g states at time t. Each site follows the same prescribed rules for up dating. For the rest of this work we will only concentrate on one-dimensional **CA**. The number of neighborhood of each site is denoted by $2k$ (In what follows k is always reserved specially for this context). The **CA** starts out with arbitrary initial configuration which is represented by $\mathbf{S(0)}=\{s_1, s_2, s_3, \cdots, s_N\}$ where s_i can be in any one of the g states. The configuration of system at time t is denoted by $\mathbf{S(t)}=\{s_1(t), s_2(t), s_3(t), \cdots, s_N(t)\}$. In this paper the state of each site is restricted to $g = 2$ such that $s_i \in \{1, -1\}$ and the updating rule is given by the *majority rule* defined as follows. The neighbors of s_i is defined as $\{s_{i+\alpha} | \alpha = \pm 1, \pm 2, \pm 3, \cdots, \pm k\}$. By introducing

$$q_i = \sum_{\alpha=-k}^{k} s_{i+\alpha}, \tag{1}$$

the updating rule can be expressed as:

$$s_i(t+1) = \begin{cases} 1 & if \quad q_i \quad > 0 \\ -1 & \qquad\qquad < 0 \\ s_i(t) & \qquad\qquad = 0 \end{cases} \tag{2}$$

It is noted that the $k = 1$ case is also known as nearest neighbor **CA**. In this work, k is arbitrary and hence our results go beyond nearest neighbor **CA**.

For finite one-dimensional **CA**, the lattice is arranged on a circle with periodic boundary conditions. Such cellular automata have a finite number of states, and as a result, after sufficient long time evolution the system must enter the state which can either be homogenous, steady or periodic state. Therefore the class 3 and 4 attractors do not exist in finite **CA**. In this paper exact results of a class of **CA** are established for *infinite extend*.

3 Lemmas and product rules

To represent the state of each site more conveniently, from now on, the binary state $\{1, -1\}$ will be graphically presented as $\{+, -\}$ respectively. The one-dimensional **CA** starts out with some initial configuration which is represented by $\mathbf{S(0)}$=$\{s_1, s_2, s_3, \cdots, s_N\}$ where each s_i can take either $+$ or $-$. For any block of size m (with $m \geq k+1$) which contains only $+(-)$ site states is denoted as $A_m(B_m)$ (In the following discussion, the subscript of a block represents the block size. m is reserved to denote block size with $m \geq k+1$.):

$$A_m = \{\underbrace{\cdots + + + \cdots}_{m}\}$$

$$B_m = \{\underbrace{\cdots - - - - \cdots}_{m}\}.$$

Given k, any block of l sites without containing more than k consecutive $+$ or $-$ site states is denoted by X_l. By using A_m, B_m and X_l, any state can be symbolically decomposed as a product of A_m, B_m and X_l, such as $\{\cdots A_{m_1} X_{l_2} A_{m_3} B_{m_4} \cdots X_{l_5} B_{m_6} \cdots\}$. For instance, for $k = 2$, the following configuration can be expressed as:

$$\begin{aligned}
&\{- - - - + - + - - + - + + + - +\} \\
&= \{- - - -\}\{+ - + - - + -\}\{+ + +\}\{- +\} \\
&= \{B_4 X_7 A_3 X_2\}
\end{aligned}$$

Obviously, this decomposition procedure is not unique. However, it is important to note that this decomposition procedure is just a tool for analyzing the state configuration at each time step and has nothing to do with the dynamics of system. During evolution, the size of each block may change and new block configurations will be generate. Our main idea of proving the existence of attractors is that *the evolution of* CA *can be reduced to the evolution of these block variables.* For the subclass of cellular automata considered in this paper, a fundamental properties which constitutes the foundation of the final proof are given as lemmas.

Lemma 1 *For* $2k$ *neighbors, the blocks* A_{k+1} *and* B_{k+1} *do not shrink during time evolution.*

The proof of this lemma can easily be seen by noting that the q_i of each site in $A_{k+1}(B_{k+1})$ is positive(negative) or equal to zero. Hence these blocks can not shrink as the system evolves. In fact the size of any A_m and B_m may grow during time evolution. This is due to the fact that all X_l configurations do not contain more than k consecutive $+$ or $-$ site states, therefore the state of boundary site of X_l that is next to $A_m(B_m)$ will switch sign as the system evolves. In either case, the size of $A_m(B_m)$ grows. By assigning a notion of parity to A_m as positive and B_m as negative, this growing process expands in both directions until both boundary sites meet another different parity block, then the size of the block remains the same in subsequent evolution. As a consequence, if the initial configuration contains either A_m or B_m, then the structure of the attractor of these **CA** depends on the evolution of X_l. From the following lemma, any finite X_l can only lead to homogeneous or steady states, hence the chaotic state can never arise:

Lemma 2 *Any finite* X_l *with either* A_m *or* B_m *as boundaries on both sides with* $m \geq k+1$ *will be eliminated during evolution.*

Since the lemma is quite obvious we omit its proof. In fact this lemma has been proved for any finite initial condition[14] which means that both A_m and B_m are of infinite size. From **Lemma 2** one can see that these requirements are not necessary. Therefore, for any initial state without any block of X_∞, the attractors of **CA** consist of the following configurations: homogeneous and steady states. For example with any k, the state

$$S = \{B_{m_1} A_{m_2} B_{m_3} A_{m_4}\},$$

is a steady state which does not change during time evolution.

From above discussions, it is necessary to see if the chaotic state can arise from any initial state containing X_∞. By introducing and analyzing the combination of products of the fundamental blocks which will be defined below, the problem of X_∞ can be reduced to a problem of finite X_l.

For any X_l with $l \geq 3(k+1)$, one can always pick a smaller section $X_{3(k+1)}$ ($\subset X_l$) and consider the evolution of this block. This smaller section can then be split into three subblocks each of which has a size of $k+1$:

$$X_{3(k+1)} = X_{k+1} Y_{k+1} Z_{k+1},$$

For example, when $k = 2$, a configuration $X_{3(k+1)} = X_9 = \{- - + - + - + + -\}$ can be decomposed as:

$$X_9 = \{- - + - + - + + -\} = \{- - +\}\{- + -\}\{+ + -\}.$$

In this way, any $X_{3(k+1)}$ block can be built up by all the possible X_{k+1}'s which are defined as fundamental blocks and we also denote them as X without the subscript. Obviously, by definition, the fundamental blocks do not contain more than k consecutive $+$ or $-$ site states. The number of fundamental blocks is $2(2^k - 1)$ and the total number of product combinations of fundamental blocks for constructing $X_{3(k+1)}$ is bounded by $8(2^k - 1)^3$ which can be classified into five types: $\{XXX\}$, $\{YXX\}$, $\{XXY\}$, $\{XYX\}$ and $\{XYZ\}$. Here, X, Y, Z represent different fundamental blocks respectively. In order to proof the existence of the proposed attractors, one has to address how these possible configurations evolve in time. It is obvious that the number of product combinations grow rapidly as k increases. However, by realizing some relations among state configurations, the number of product combinations needed to be considered can be reduced effectively.

It is easy to check that the product combinations of $\{XXY\}$, $\{YXX\}$ and $\{XXX\}$ will produce A_m or B_m which will not shrink in subsequent evolution. For example, for $k = 2$, the product of two fundamental blocks $\{- + -\}\{- + -\}$ will evolve to a configuration containing a block of B_4 in the next time step. From **Lemma 1** and **2**, one can conclude that $\{-+-\}\{-+-\}$ can only contribute to the transient states. This situation will be addressed in detail for the case of $k = 2$ in the next section. Therefore the cases required further investigation are the product combinations of $\{XYX\}$ and $\{XYZ\}$ types.

As mentioned before, there exist relations among state configuration in one-dimensional **CA** such that the evolution of the product combinations can be simplified. The simplification is achieved by using two types of operations which were pointed out in [4]. For any configuration $\mathrm{S(t)}=\{\cdots s_{n-1}, s_n, s_{n+1}, \cdots\}$, the conjugate state of $\mathrm{S(t)}$ is denoted as $\widetilde{\mathrm{S}}(t)$ defined by flipping all the state of s_i. For example, the state $\mathrm{S(0)} = \{- + + - +-\}$ has a corresponding conjugate state $\widetilde{\mathrm{S}}(0) = \{+ - - + -+\}$. Moreover, the state $\mathrm{S(t)}$ also has an image state associate to it. The image state $\bar{\mathrm{S}}(t)$ of $\mathrm{S(t)}$ is defined by reversing sequence of the states of sites of $\mathrm{S(t)}$. That is to say, for any $\mathrm{S(t)}$ with N sites, $S(t) = \{s_1, s_2, \cdots, s_{N-2}, s_{N-1}, s_N\}$, the associate image state is

$$\begin{aligned}
\bar{\mathrm{S}}(t) &= \{\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{N-2}, \bar{s}_{N-1}, \bar{s}_N\} \\
&= \{s_N, s_{N-1}, s_{N-2}, \cdots, s_2, s_1\}.
\end{aligned}$$

For the $\mathrm{S(0)}$ considered above, the corresponding $\bar{\mathrm{S}}(0)$ is $\{-+-++-\}$. It is important to realize that these operations are preserved under time evolution. These results are stated as theorems:

Theorem 1 *If* $\widetilde{\mathrm{S}}(t)$ *is the conjugate state of* $\mathrm{S(t)}$*, then* $\widetilde{\mathrm{S}}(t+1)$ *is the conjugate state of* $\mathrm{S(t+1)}$.

Proof: Under time evolution

$$s_i(t+1) = sgn\{\sum_{\alpha=1}^{k} (s_{i+\alpha} + s_{i-\alpha})\},$$

where sgn is the conventional sign-function and if $\sum(s_{i+\alpha} + s_{i-\alpha}) = 0$, then $s_i(t+1) = s_i(t)$. The evolution of $\widetilde{\mathbf{S}}(t)$ is given by:

$$\begin{aligned}
\tilde{s}_i(t+1) &= sgn\{\sum_{\alpha=1}^{k}[\tilde{s}_{i+\alpha}(t) + \tilde{s}_{i-\alpha}(t)]\} \\
&= sgn\{\sum_{\alpha=1}^{k}[(-1)s_{i+\alpha}(t) + (-1)s_{i-\alpha}(t)]\} \\
&= sgn\{(-1)\sum_{\alpha=1}^{k}[s_{i+\alpha}(t) + s_{i-\alpha}(t)]\} \\
&= (-1)s_i(t+1).
\end{aligned}$$

When $\sum(s_{i+\alpha} + s_{i-\alpha}) = 0$, $s_i(t+1) = s_i(t)$, then

$$\tilde{s}_i(t+1) = (-1)s_i(t) = \tilde{s}_i(t).$$

That is to say, if $s_i(t)$ remains unchanged, so does $\tilde{s}_i(t)$. The proof is complete.

Theorem 2 *If* $\bar{\mathbf{S}}(t)$ *is the image state of* $\mathbf{S}(\mathbf{t})$, *then* $\bar{\mathbf{S}}(t+1)$ *is the image state of* $\mathbf{S}(\mathbf{t}+1)$.

Proof: The proof is similar to the **theorem 1**. The image of $S(t)$ is given by

$$\bar{S}(t) = \{\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{N-2}, \bar{s}_{N-1}, \bar{s}_N\}$$

with $\bar{s}_i = s_{N-i}$. At the next time step, one has

$$\begin{aligned}
\bar{s}_i(t+1) &= sgn\{\sum_{\alpha=1}^{k}[\bar{s}_{i+\alpha}(t) + \bar{s}_{i-\alpha}(t)]\} \\
&= sgn\{\sum_{\alpha=1}^{k}[s_{N-(i+\alpha)}(t) + s_{N-(i-\alpha)}(t)]\} \\
&= sgn\{\sum_{\alpha=1}^{k}[s_{(N-i)+\alpha}(t) + s_{(N-i)-\alpha}(t)]\} \\
&= s_{N-i}(t+1).
\end{aligned}$$

It is also true that if $s_{N-i}(t+1) = s_{N-i}(t)$, then $\bar{s}_i(t+1) = \bar{s}_i(t)$. Thus the image operation is persevered under time evolution.

The method of establishing the existence of steady states is to show that all $X_{3(k+1)}$ can produce A_m or B_m as the system evolves. Once A_m or B_m occurred, by using **lemma 1** and **2**, the steady state is obtained. From now on A_m and B_m will be referred as the *uniform blocks*. However, one of the important properties of the conjugation operation is that the blocks A_{m_a} and B_{m_b} with $m_a = m_b$ form a conjugate pair. This fact is very useful in simplifying the analysis of the attractors.

It is important to realize that the conjugation procedure divides the set of all possible product combinations of $X_{3(k+1)}$ into two mutually conjugate classes. Any $X_{3(k+1)}$ will belong to one of these two classes unless it is self-conjugate. The mutual conjugation between A_{m_a} and B_{m_a} reduces the analysis on just one class. This reduction can be seen from the fact that if $Q_{3(k+1)}$ which is in either of classes produces a block A_{m_a} during evolution, then by applying **theorem 1**, the conjugate block $\tilde{Q}_{3(k+1)}$ which belongs to the other conjugate-class will produce B_{m_a}. As a

result, only one of the classes denoted by $\mathbf{\Gamma}$ requires further discussion. (The other conjugate-class is denoted by $\tilde{\mathbf{\Gamma}}$.) On $\mathbf{\Gamma}$, further reduction can also be obtained by considering the imaging operation. By considering two product combinations in $\mathbf{\Gamma}$ which form the image pair, if one of the pair produces either of the uniform blocks during evolution, then the other one results in a corresponding block respectively. Hence, one of these two product combinations can be eliminated without losing generality. However, in $\mathbf{\Gamma}$, there are some product combinations that the image of them appear in $\tilde{\mathbf{\Gamma}}$. Then the conjugation of these image product combinations are also in $\mathbf{\Gamma}$ and can also be eliminated. This fact can easily be seen by using the combined results of **theorems 1** and **2**. The subset of $\mathbf{\Gamma}$ obtained after the above reductions is the set of product combinations which require detail analysis of the product rules. In the next section a detail discussion on the product rules for $\boldsymbol{k=2}$ will be presented.

4 The $\boldsymbol{k = 2}$ Model

For $\boldsymbol{k=2}$, there are six fundamental blocks:

$$\begin{array}{lll} \boldsymbol{E} = \{+-+\} & \boldsymbol{G} = \{++-\} & \boldsymbol{H} = \{-++\} \\ \boldsymbol{F} = \{-+-\} & \boldsymbol{I} = \{--+\} & \boldsymbol{J} = \{+--\}, \end{array} \tag{3}$$

From (3), it is obvious that

$$\{\boldsymbol{E}, \boldsymbol{G}, \boldsymbol{H}\} \tag{4}$$

are the conjugate-blocks of $\{\boldsymbol{F}, \boldsymbol{I}, \boldsymbol{J}\}$ respectively. Moreover, the image operation produces the image pairs, $(\boldsymbol{G}, \boldsymbol{H})$ and $(\boldsymbol{I}, \boldsymbol{J})$. All $\boldsymbol{X_9}$ can be built up by the product combinations of (3) and altogether one has at most **216** product combinations. In considering the evolution of any $\boldsymbol{X_9}$, as mentioned before, it can be checked explicitly that the product combinations of $\{\boldsymbol{XXX}\}$, $\{\boldsymbol{XXY}\}$ and $\{\boldsymbol{YXX}\}$ types can produce the uniform block in the next time step and lead to the steady state. Note that this fact is also true for arbitrary $\boldsymbol{k}$. Therefore, the main goal of the following discussion is to prove the existence attractors generated from $\{\boldsymbol{XYX}\}$ and $\{\boldsymbol{XYZ}\}$.

Let us start from the reduction procedures. It is noted that there are some products where $\boldsymbol{A_m}$ or $\boldsymbol{B_m}$ arises automatically as two fundamental blocks are grouped together. These products are excluded from the initial state with configuration $\boldsymbol{X_\infty}$. For example, the product of $\boldsymbol{E}$ and $\boldsymbol{G}$ is

$$\{\boldsymbol{EG}\} = \{+-+++-\} = \{\boldsymbol{X_2 A_3 X_1}\}.$$

The complete list of these particular results is as follows:

$$\{\boldsymbol{EG}, \boldsymbol{FI}, \boldsymbol{GI}, \boldsymbol{IG}, \boldsymbol{HG}, \boldsymbol{JI}, \boldsymbol{HJ}, \boldsymbol{JH}, \boldsymbol{HE}, \boldsymbol{JF}\}. \tag{5}$$

Moreover, there are also four products which can evolve to uniform blocks within one step and without referring to their neighboring fundamental block. These products are $\{\boldsymbol{EH}, \boldsymbol{GE}, \boldsymbol{FJ}, \boldsymbol{IF}\}$, and will also be omitted in the following discussion. As a result, the remaining products of two fundamental blocks are:

$$\{\boldsymbol{EF}, \boldsymbol{EI}, \boldsymbol{EJ}, \boldsymbol{GF}, \boldsymbol{GH}, \boldsymbol{GJ}, \boldsymbol{HF}, \boldsymbol{HI}, \boldsymbol{FE}, \boldsymbol{FG}, \boldsymbol{FH}, \boldsymbol{IE}, \boldsymbol{IJ}, \boldsymbol{IH}, \boldsymbol{JE}, \boldsymbol{JG}\}. \tag{6}$$

This set can be divided into two subsets by conjugation relation. They are symbolically presented as follows for expressing the product relation more effectively:

$$\boldsymbol{E} \mapsto \left\{ \begin{array}{c} \boldsymbol{F} \\ \boldsymbol{I} \\ \boldsymbol{J} \end{array} \right\}, \boldsymbol{G} \mapsto \left\{ \begin{array}{c} \boldsymbol{F} \\ \boldsymbol{H} \\ \boldsymbol{J} \end{array} \right\}, \boldsymbol{H} \mapsto \left\{ \begin{array}{c} \boldsymbol{F} \\ \boldsymbol{I} \end{array} \right\} \tag{7}$$

and

$$\boldsymbol{F} \mapsto \left\{ \begin{array}{c} \boldsymbol{E} \\ \boldsymbol{G} \\ \boldsymbol{H} \end{array} \right\}, \boldsymbol{I} \mapsto \left\{ \begin{array}{c} \boldsymbol{E} \\ \boldsymbol{J} \\ \boldsymbol{H} \end{array} \right\}, \boldsymbol{J} \mapsto \left\{ \begin{array}{c} \boldsymbol{E} \\ \boldsymbol{G} \end{array} \right\}. \tag{8}$$

Therefore, if the product combinations generated from (7) are in the class $\boldsymbol{\Gamma}$, then the ones generated from (8) must belong to the corresponding conjugate-class $\tilde{\boldsymbol{\Gamma}}$. Keeping these products and the conjugate relation between (7) and (8) in mind, the structure of attractors can be shown explicitly.

Starting from $\{\boldsymbol{XYX}\}$, because of the conjugate relation between (7) and (8), one only need to use the group (7) to construct the product combinations of such type. By further considering the imaging operations, the product combinations can be reduced to only **4** cases:

$$\{\boldsymbol{EFE}\}, \{\boldsymbol{EIE}\}, \{\boldsymbol{GFG}\}, \{\boldsymbol{GJG}\}. \tag{9}$$

These combinations by itself do not contain enough information to justify that the system will produce uniform blocks at the next time step. However, it will be shown latter that these product combinations can lead to either the steady or periodic states.

For the $\{\boldsymbol{XYZ}\}$ type, the analysis can be proceeded by concentrating on the ones in $\boldsymbol{\Gamma}$. There are **14** product combinations required further consideration. By using the imaging operation for further reduction, the number of remaining product combinations is **10**. Moreover, by combining the procedures of conjugation and imaging together, only **8** combinations remain:

$$\{\boldsymbol{EFG}\}, \{\boldsymbol{EFH}\}, \{\boldsymbol{EIJ}\}, \{\boldsymbol{EIH}\}, \{\boldsymbol{EJG}\}, \{\boldsymbol{GFH}\}, \{\boldsymbol{GHI}\}, \{\boldsymbol{HFG}\}. \tag{10}$$

It is easy to check that except for the case of $\{\boldsymbol{GHI}\}$, the rest of (10) will generate uniform block within one or two steps. For $\{\boldsymbol{GHI}\}$, in order to check whether it can generate uniform blocks, it is necessary to take into account the effects of its neighboring blocks as the system evolves. Since $\{\boldsymbol{EG}, \boldsymbol{IG}, \boldsymbol{HG}\}$ is a subset of (5) which already contain the uniform blocks, the neighboring blocks of the leftmost side of $\{\boldsymbol{GHI}\}$ should not include any of $\{\boldsymbol{E}, \boldsymbol{I}, \boldsymbol{H}\}$. Similarly, for the rightmost side of $\{\boldsymbol{GHI}\}$, one should not include any of $\{\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{J}\}$. Thus, the following combinations require separate investigation:

$$\{\boldsymbol{FGHIE}\}, \{\boldsymbol{FGHIH}\}, \{\boldsymbol{FGHIJ}\}, \{\boldsymbol{JGHIE}\}, \{\boldsymbol{JGHIH}\}, \{\boldsymbol{JGHIJ}\}. \tag{11}$$

By using the fact that the conjugate of $\{\boldsymbol{FGH}\}$ is $\{\boldsymbol{EIJ}\}$ which is already in (10), that is to say any block in (11) which contains $\{\boldsymbol{FGH}\}$ will generate the uniform block. This discussion also apply to $\{\boldsymbol{JGHIE}\}$ and $\{\boldsymbol{JGHIH}\}$. However, the remaining case $\{\boldsymbol{JGHIJ}\}$ does not produce the uniform block. This particular block suggests a periodic configuration. In fact the infinite sequence of $\{\boldsymbol{JGHI}\}$ is a periodic state:

$$\begin{array}{c} \{\cdots \boldsymbol{JGHIJGHI} \cdots\} \\ \downarrow \\ \{\cdots \boldsymbol{HIJGHIJG} \cdots\} \\ \downarrow \\ \{\cdots \boldsymbol{JGHIJGHI} \cdots\} \end{array}$$

Furthermore, by checking explicitly, any other neighboring block attached to $\{\boldsymbol{JGHIJ}\}$ always create a uniform block within two steps.

Similar analysis can now be applied for analyzing the product combinations in (9). This can be illustrated by the case of $\{\boldsymbol{EFE}\}$. According to the rules given by (7) and (8), the only neighboring blocks of $\{\boldsymbol{EFE}\}$ which do not generate uniform blocks is the infinite sequence: $\{\cdots \boldsymbol{FEFEFEFE} \cdots\}$. This configuration is a steady state. By taking this block configuration with infinite sequence, the remaining product combinations in (9) are periodic states. For instance, considering $\{\boldsymbol{EIE}\}$, the infinite sequence is $\{\cdots \boldsymbol{EIEIEI} \cdots\}$. Under time evolution, this state oscillates:

$$\begin{array}{c}\{\cdots \boldsymbol{EIEIEI}\cdots\}\\ \downarrow \\ \{\cdots \boldsymbol{GFGFGF}\cdots\}\\ \downarrow \\ \{\cdots \boldsymbol{EIEIEI}\cdots\}\end{array}$$

The other periodic state is:

$$\{\cdots \boldsymbol{GJGJGJ}\cdots\} \Longleftrightarrow \{\cdots \boldsymbol{IHIHIH}\cdots\}$$

On the other hand, any other neighboring block attached to any blocks in (9) will produce uniform blocks by checking directly. In passing we note that the periodic attractors are period two as the finite **CA**.

To summarize, for $\boldsymbol{k = 2}$, from above analysis, we reduce the number of product combinations from **216** down to **12** ((9)+(10)) and it is shown that all attractors of the $\boldsymbol{k = 2}$ model are consisted of three different types:

1. The spatially homogeneous states such as $\boldsymbol{A_\infty}$ and $\boldsymbol{B_\infty}$.
2. The steady states of the form $\boldsymbol{\{...EFEF...\}}$ and $\boldsymbol{\{...A_{m_i}B_{m_j}A_{m_p}B_{m_q}...\}}$ with $\forall \boldsymbol{m_a} \geq \boldsymbol{k+1}$.
3. The periodic states which includes $\boldsymbol{\{...JGHIJGHI...\}}$, $\boldsymbol{\{...EIEI...\}}$ and $\boldsymbol{\{...GJGJ...\}}$.

Therefore, the chaotic state and complex structure state do not exist in this cellular automaton.

5 Discussions and conclusions

By realizing fundamental blocks structure of the **CA** model with majority rule, it is shown in this paper that this model has certain nice properties which can be used to establish the structure of the attractors. It is shown explicitly that, with $\boldsymbol{k = 2}$, for arbitrary initial configuration the only allowed attractors are the spatially homogeneous, steady and periodic states. Thus, the chaotic state and complex structure state referred in Wolfram's classification [3] are excluded. Our method also provides explicit enumeration of all periodic states which are in fact period two as in the case of finite lattice [13].

Although the proof was established for $\boldsymbol{k = 2}$, this approach is general enough for showing the result of arbitrary $\boldsymbol{k}$. The approach is based on the fact that the uniform blocks $\boldsymbol{A_{k+1}}$ and $\boldsymbol{B_{k+1}}$ grow during evolution. Furthermore, by using the fact that any initial condition can be expressed as block products, therefore the problem of deriving the attractors turns into the problem of showing that any $\boldsymbol{X_{3(k+1)}}$ can generate the uniform blocks. Even though the number of product combinations of $\boldsymbol{X_{3(k+1)}}$ increases with $\boldsymbol{k}$, by using the conjugation and imaging procedures, the amount of explicit checking on the product analysis is greatly reduced. This approach has also been done for the case of $\boldsymbol{k = 3}$ but the analysis is too lengthy to be presented here and hence omitted. Thus, this work provides a complete analysis of the attractors of infinite **CA** with majority rules. Our results hold for any initial configuration in contrast to the results of previous works which assume either finite initial states or finite lattice. One can conclude that for this subclass of **CA** model, the only allowed attractors are homogenous, steady and periodic states.

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